CONTROLLER DESIGN IN TIME-DOMAIN

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Keywords: Tracking control, Disturbance rejection, Performance specifications, Integral criteria, Optimal controller setting, Tuning rules for standard controllers, Empirical design, Standard polynomials.

Contents

Summary

This article presents an introduction to the classical design methods for linear continuous time-invariant single input/single output control systems in the time-domain. The design is based on finding the "best" possible controller with respect to selected time-domain performance specifications. For the dynamic behavior of the closed-loop control system, performance specifications are defined for the input step responses of the reference signal and disturbance. These transient performance specifications are natural and are used to formulate the desired closed-loop behavior of the control system. However, these specifications are more appropriate for evaluating the result of a control system design, whereas the design is usually based on minimizing specific integral performance indices using various functions of the error between the reference input and the controlled plant output. Especially in the case of a fixed controller structure, these integral criteria provide optimal controller settings. The solution of this optimization problem can be obtained by numerical or analytical approaches. In the time-domain design, empirical procedures, such as tuning rules for standard controllers or design by computer simulation play an

important role. Time-domain specifications can also be used to select standard-polynomials, such as the characteristic polynomial for the desired closed-loop transient behavior. This leads to a mixed time- and frequency-domain design, where the solution provides the structure and parameters of the controller as a result of the selected time-domain performance specification.

1. Problem Formulation

The design of a control system may lead to different solutions to meet explicit design goals, but also implicit engineering goals such as economical considerations, complexity and reliability. The design procedure depends on whether the nominal plant transfer function $G_\mathrm{P}(s)$ is known or not.

In any case, the "best" possible controller or compensator transfer function $G_\mathrm{C}(s)$ has to be designed or selected and tuned such that the desired performance specifications are met. In general the designed closed-loop system, considered in Figure 1, should at least fulfil the following conditions:

1) The closed-loop system has to be stable.
2) Disturbances $d(t)$ should have only a minimal influence on the controlled variable $y(t)$.
3) The controlled variable $y(t)$ must be able to track the reference signal $r(t)$ as fast and as accurately as possible.
4) The closed-loop system should not be too sensitive to parameter changes of the plant.

In order to fulfil conditions 2) and 3) the closed-loop transfer function for tracking control in the *ideal* case should be, assuming unity feedback

$$G_\mathrm{R}(s) = \frac{Y(s)}{R(s)} = \frac{G_0(s)}{1 + G_0(s)} = 1, \qquad (1)$$

where $G_0(s) = G_\mathrm{C}(s)\, G_\mathrm{P}(s)$ is the open-loop transfer function, and the corresponding ideal transfer function for the closed-loop in the case of disturbance rejection should be

$$G_\mathrm{D}(s) = \frac{Y(s)}{D(s)} = \frac{1}{1 + G_0(s)} = 0 \qquad (2)$$

Theoretically, Eqs. (1) and (2) can only be satisfied if $G_0(s) >> 1 \;\forall\, s$, which will be the case for a large value of the gain factor $K_0 >> 1$ of $G_0(s)$, where K_0 is the gain factor of $G_0(s)$. It should be noted that in this article only unity feedback is considered. The addition of a feedback controller can enhance stability and design flexibility.

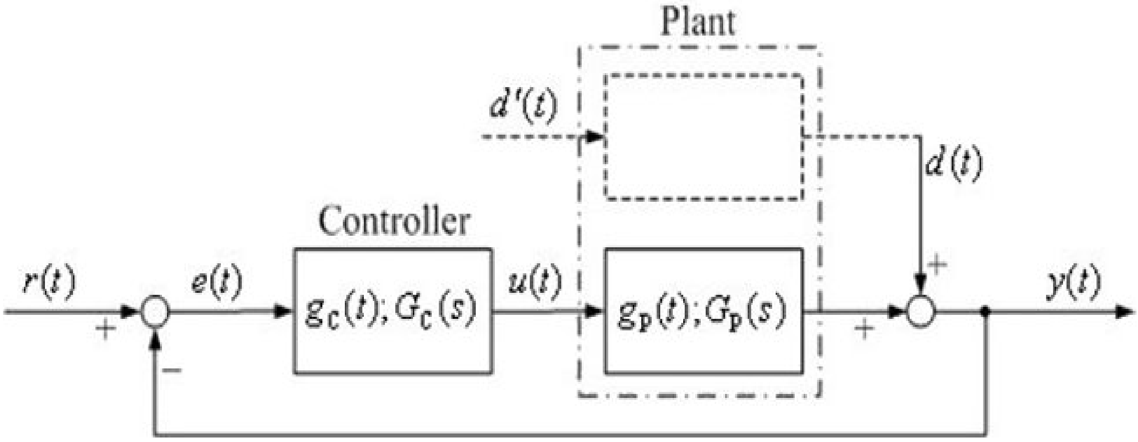

Figure 1. Block diagram of a standard linear closed-loop control system

However, both conditions, Eqs. (1) and (2), cannot be satisfied strictly, due to physical limitations especially concerning the controller gain and the magnitude of the manipulating signal. Furthermore, increasing K_0 too much would lead in most cases to stability problems. In practice, the design engineer has to make a compromise between the desired behavior and the technical limitations.

This procedure needs a lot of experience, and engineering judgement, as well as intuition. Thus, it is understandable that for the design of control systems either in the frequency- or time-domain many different approaches are available and provide different solutions. Each solution is optimal with respect to the selected measure of performance. In this article only some classical design methods in the time-domain are considered. The design of state feedback controllers is, therefore, discussed separately (see *Design of State Space Controllers for SISO Systems*).

2. Time-Domain Performance Specifications

The starting point for the design of a feedback-control system is to have a good plant model described either in the form of a differential equation or a transfer function $G_P(s)$. Once the plant model is given, the next step is to design an overall system, as shown in Figure 1, that meets the desired design specifications.

It is important to note that different applications may require different specifications. Generally, the performance of feedback-control systems includes two tasks: steady-state performance, which specifies accuracy when all the transients are decayed (see *Closed-loop Behvior*), and transient performance, which specifies the speed of response as discussed below.

2.1. Transient Performance

The transient performance is usually defined for a step reference or step disturbance input response as shown in Figure 2. The specifications indicated in Figure 2 are natural. In the case of reference tracking (see Figure 2a) these specifications are as follows:

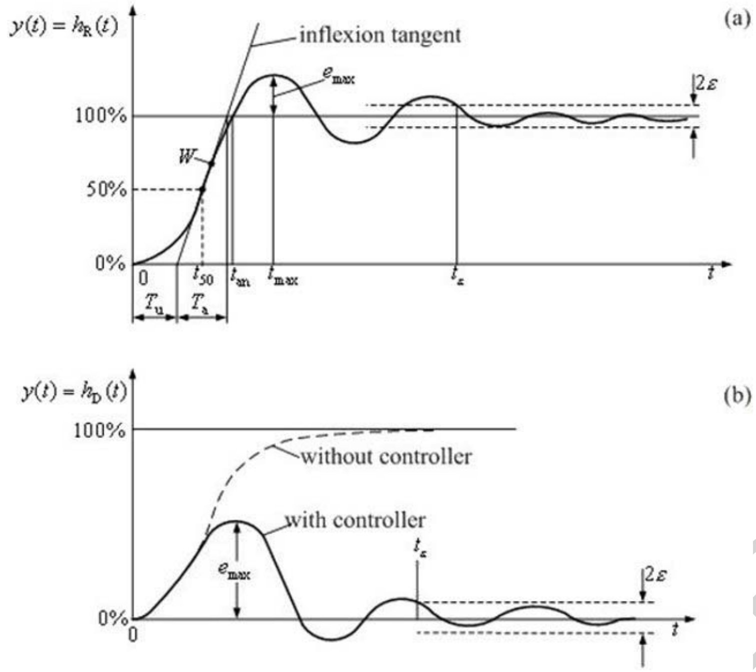

Figure 2. Step responses for (a) reference input and (b) disturbance input including main parameters of transient performance

Peak overshoot e_{max} : This term is defined as the maximum value of the response at time t_{max} in relation to its desired final value. It can be considered to be a measure of the relative stability of the system. It increases as the damping ratio decreases.

Rise time T_{a} : Is defined often as the time required for a response to go from 10 % to 90 % of its desired final value, or as the time interval given by the intersection points of the inflexion tangent with the 0 % and 100 % lines.

Delay time T_{u} : This is the time between the excitation and the intersection point of the inflexion tangent of the response with the 0 % line.

Settling time t_{ε} : This term is the time after which the response remains within a band of $\pm\,\varepsilon\ \%$ about the desired final value, where ε is selected between 2 % and 5 %.

Reaching time t_{an} : This is the time at which the response reaches for the first time the desired final value, where $t_{\text{an}} \approx T_{\text{u}} + T_{\text{a}}$.

Similarly, the case of disturbance rejection (see Figure 2b) can be characterized by introducing the peak overshoot and settling time. Whereas e_{max} and t_{ε} depend upon the damping ratio, the other values T_{a} , t_{max} and t_{an} represent a measure for the speed of the transient behavior.

2.2. Integral Criteria

The performance specifications introduced above are appropriate for evaluating the results of a control system design, however, they cannot be used directly as a starting point for designing a controller. It would be desirable to have criteria based on only one factor, for example,

$$I = k_1 t_{\mathrm{an}} + k_2 t_\varepsilon + k_3 e_{\max}, \tag{3}$$

where $k_i (i = 1, 2, 3)$ are weighting factors characterizing the relative importance of each of the performance specifications. The design of a controller leading to the smallest value of I is called optimal in the sense of this criterion. However, the individual selection of the k_i-factors and the analytical evaluation of Eq. (3) usually causes difficulties. Therefore, performance indices based on various functions $f_k[e(t)]$ of the error

$$e(t) = y(t) - r(t) \tag{4}$$

between the reference input $r(t)$ and the controlled plant output $y(t)$ are preferred. General performance indices covering an error function in $[0, \infty)$ have been introduced as the integral

$$I_k = \int_0^\infty f_k[e(t)]\,\mathrm{d}t, \tag{5}$$

where $f_k\,[e(t)]$ can take various forms as shown in Table 1.

Having defined various performance indices according to Table 1, the *integral criteria* can be formulated as follows: A closed-loop control system is optimal subject to the selected performance index I_k if the adjustable controller settings $r_1, r_2, \ldots$ or the controller structure are selected such that I_k becomes minimal:

$$I_k = \int_0^\infty f_k[e(t)]\,\mathrm{d}t = I_k(r_1, r_2, \ldots) \overset{!}{=} \mathrm{Min}. \tag{6}$$

Performance Index	**Characteristics**
$I_1 = \int_0^\infty e(t)\,\mathrm{d}t$	*Integral of total error* (ITE): Only appropriate for highly damped monotonic step responses of $e(t)$; simple mathematical treatment.
$I_2 = \int_0^\infty \lvert e(t)\rvert\,\mathrm{d}t$	*Integral of absolute error* (IAE): Appropriate for non-monotonic step responses. Not easy to track analytically.

$I_3 = \int_0^\infty e^2(t)\,\mathrm{d}t$	*Integral of square error* (ISE): Penalizes large errors more heavily than small ones; provides longer t_ε as I_2. In many cases analytical tracking is possible.
$I_4 = \int_0^\infty \lvert e(t)\rvert\, t\,\mathrm{d}t$	*Integral of time multiplied absolute error* (ITAE): Provides similar results as I_2; puts less weight on $e(t)$ for *t* small and more for *t* large.
$I_5 = \int_0^\infty e^2(t) t\,\mathrm{d}t$	*Integral of time multiplied square error* (ITSE): Provides similar results as combined with the same time weighting as for I_4
$I_6 = \int_0^\infty [e^2(t) + \alpha \dot{e}^2(t)]\,\mathrm{d}t$	Integral of generalized square error (IGSE): Better results as for I_3 are obtained, however, the selection of the weighting factor α is subjective.
$I_7 = \int_0^\infty [e^2(t) + \beta u^2(t)]\,\mathrm{d}t$	*Integral of square error and control effort* (ISECE): Provides a slightly larger $e_{\max}$, but t_ε becomes essentially smaller as for ; however, the selection of β is subjective.

Table 1. Various integral performance indices (Note: If the closed-loop systems has a steady-state error e_∞, then $e(t)$ must be replaced by $e(t) - e_\infty$)

The minimum of I_k may be located inside or, due to constraints, on the boundary of the parameter space, whose coordinates are defined by the adjustable controller parameters $r_i\,(i = 1, 2, \ldots)$. Both cases lead to different mathematical treatments. In the first case an *absolute* or global optimum of I_k is obtained, whereas in the second case a *boundary* or relative optimum occurs.

2.3. Calculation of the ISE-Performance Index

In many cases, the criterion based on minimal ISE (integral of the squared error) - performance index (I_3 in Table 1) is appropriate. Furthermore, the analytical treatment of the most important cases is possible. The calculation of this performance index is based on Parseval's theorem,

$$I_3 = \int_0^\infty e^2(t)\,\mathrm{d}t = \frac{1}{2\pi\,\mathrm{j}} \int_{-\mathrm{j}\infty}^{+\mathrm{j}\infty} E(s)E(-s)\,\mathrm{d}s\,, \tag{7}$$

where $E(s)$ is the Laplace transform of $e(t)$ and is assumed to be a fractional rational function

$$E(s) = \frac{c_0 + c_1 s + \ldots + c_{n-1}s^{n-1}}{d_0 + d_1 s + \ldots + d_n s^n}\,. \tag{8}$$

If all the poles of $E(s)$ are located in the left-hand side (LHS) of the complex *s*-plane, then Eq. (7) converges and can be solved by partial-fraction expansion. For *n* up to 10 the values of I_3 exist in a tabular form. Table 2 contains the integrals for *n* up to 4.

$I_{3,1} = \dfrac{c_0^2}{2d_0 d_1}$
$I_{3,2} = \dfrac{c_1^2 d_0 + c_0^2 d_2}{2d_0 d_1 d_2}$
$I_{3,3} = \dfrac{(c_1^2 d_0 d_1 + (c_1^2 - 2c_0 c_2) d_0 d_3 + c_0^2 d_2 d_3)}{2d_0 d_3 (-d_0 d_3 + d_1 d_2)}$
$I_{3,4} = \dfrac{(c_3^2 (-d_0^2 d_3 + d_0 d_1 d_2) + (c_2^2 - 2c_1 c_3) d_0 d_1 d_4 + (c_1^2 - 2c_0 c_2) d_0 d_3 d_4 + c_0^2 (-d_1 d_4^2 + d_2 d_3 d_4))}{2d_0 d_4 (-d_0 d_3^2 - d_1^2 d_4 + d_1 d_2 d_3)}$

Table 2. ISE- performance index $I_{3,n}$ for $n = 1, 2, 3, 4$

3. Optimal Controller Settings Subject to the ISE-Criterion

For a given reference signal $r(t)$ or disturbance signal $d(t)$ the ISE-performance index I_3 according to Eq. (7) is a function $I_3(r_1, r_2, \ldots)$ depending on the adjustable controller parameters $r_i\,(i = 1, 2, \ldots)$ alone. Let $r_{i\text{opt}}$ denote the optimal controller parameters corresponding to the minimal value of I_3. The solution of this simple mathematical optimization problem,

$$I_3(r_1, r_2, \ldots) \overset{!}{=} \text{Min}\,, \tag{9}$$

is obtained by setting the partial derivatives of I_3 to zero:

$$\left.\frac{\partial I_3}{\partial r_1}\right|_{r_{2\text{opt}},\, r_{3\text{opt},\ldots}} = 0, \qquad \left.\frac{\partial I_3}{\partial r_2}\right|_{r_{1\text{opt}},\, r_{3\text{opt},\ldots}} = 0, \ldots \tag{10}$$

The set of optimal controller parameters, resulting from Eq. (10), represents the minimum of I_3 that is always located inside the stable region of the parameter space given by the coordinates r_i. If several points fulfil Eq. (10), then eventually the second derivative of I_3 must be calculated in order to check whether the extremal point represents a minimum. For the case of several local minima, the absolute minimum provides the optimal controller parameters $r_i = r_{i\text{opt}} (i = 1, 2, \ldots)$.

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Biographical Sketch

Heinz Unbehauen is Professor Emeritus at the Faculty of Electrical Engineering and Information Sciences at Ruhr-University, Bochum, Germany. He received the Dipl.-Ing. degree from the University of Stuttgart, Germany, in 1961 and the Dr.-Ing. and Dr.-Ing. habil. degrees in Automatic Control from the same university in 1964 and 1969, respectively. In 1969, he was awarded the title of Docent and in 1972, he was appointed as Professor of control engineering in the Department of Energy Systems at the University of Stuttgart. Since 1975, he has been Professor at Ruhr-University of Bochum, Faculty of Electrical Engineering, where he was head of the Control Engineering Laboratory until February 2001. He was Dean of his faculty in 1978/79. He was a Visiting Professor in Japan, India, China and the USA. He has authored and co-authored over 400 journal articles, conference papers and 7 books. He has delivered many invited lectures and special courses at universities and companies around the world. His main research interests are in the fields of system identification, adaptive control, robust control and process control of multivariable systems. He is Honorary Editor of IEE Proceedings on Control Theory and Application and System Science, Associate Editor of Automatica and serves on the Editorial Board of the International Journal of Adaptive Control and Signal Processing, Optimal Control Applications and Methods (OCAM) and Systems Science. He also served as associate editor of IEEE-Transactions on Circuits and Systems as well as Control-Theory and Advanced Technology (C-TAT). He is also an Honorary Professor of Tongji University Shanghai. He has been a consultant for many companies as well as for public organisations, e.g., UNIDO and UNESCO. He is a member of several national and international professional organisations and a Fellow of IEEE.